

Radix Sorting With No Extra Space

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Abstract

It is well known that n integers in the range $[1, n^c]$ can be sorted in $O(n)$ time in the RAM model using radix sorting. More generally, integers in any range $[1, U]$ can be sorted in $O(n\sqrt{\log \log n})$ time [5]. However, these algorithms use $O(n)$ words of extra memory. Is this necessary?

We present a simple, stable, integer sorting algorithm for words of size $O(\log n)$, which works in $O(n)$ time and uses only $O(1)$ words of extra memory on a RAM model. This is the integer sorting case most useful in practice. We extend this result with same bounds to the case when the keys are read-only, which is of theoretical interest. Another interesting question is the case of arbitrary c . Here we present a black-box transformation from any RAM sorting algorithm to a sorting algorithm which uses only $O(1)$ extra space and has the same running time. This settles the complexity of in-place sorting in terms of the complexity of sorting.

1 Introduction

Given n integer *keys* $S[1 \dots n]$ each in the range $[1, n]$, they can be sorted in $O(n)$ time using $O(n)$ space by *bucket sorting*. This can be extended to the case when the keys are in the range $[1, n^c]$ for some positive constant c by *radix sorting* that uses repeated bucket sorting with $O(n)$ ranged keys. The crucial point is to do each bucket sorting *stably*, that is, if positions i and j , $i < j$, had the same key k , then the copy of k from position i appears before that from position j in the final sorted order. Radix sorting takes $O(cn)$ time and $O(n)$ space. More generally, RAM sorting with integers in the range $[1, U]$ is a much-studied problem. Currently, the best known bound is the randomized algorithm in [5] that takes $O(n\sqrt{\log \log n})$ time, and the deterministic algorithm in [1] that takes $O(n \log \log n)$ time. These algorithms also use $O(n)$ words of extra memory in addition to the input.

We ask a basic question: *do we need $O(n)$ auxiliary space for integer sorting?* The ultimate goal would be to design *in-place* algorithms for integer sorting that uses only $O(1)$ extra words. This question has been explored in depth for comparison-based sorting, and after a series of papers, we now know that in-place, stable comparison-based sorting can be done in $O(n \log n)$ time [10]. Some very nice algorithmic techniques have been developed in this quest. However, no such results are known for the integer sorting case. Integer sorting is used as a subroutine in a number of algorithms that deal with trees and graphs, including, in particular, sorting the transitions of a finite state machine. Indeed, the problem arose in that context for us. In these applications, it is useful if one can sort in-place in $O(n)$ time. From a theoretical perspective, it is likewise interesting to know if the progress in RAM sorting, including [3, 1, 5], really needs extra space.

Our results are in-place algorithms for integer sorting. Taken together, these results solve much of the issues with space efficiency of integer sorting problems. In particular, our contributions are threefold.

A practical algorithm. In Section 2, we present a stable integer sorting algorithm for $O(\log n)$ sized words that takes $O(n)$ time and uses only $O(1)$ extra words.

This algorithm is a simple and practical replacement to radix sort. In the numerous applications where radix sorting is used, this algorithm can be used to improve the space usage from $O(n)$ to only $O(1)$ extra words. We have implemented the algorithm with positive results.

One key idea of the algorithm is to compress a portion of the input, modifying the keys. The space thus made free is used as extra space for sorting the remainder of the input.

Read-only keys. It is theoretically interesting if integer sorting can be performed in-place *without modifying the keys*. The algorithm above does not satisfy this constraint. In Section 3, we present a more sophisticated algorithm that still takes linear time and uses only $O(1)$ extra words without modifying the keys. In contrast to the previous algorithm, we cannot create space for ourselves by compressing keys. Instead, we introduce a new technique of *pseudo pointers* which we believe will find applications in other succinct data structure problems. The technique is based on keeping a set of distinct keys as a pool of preset read-only pointers in order to maintain linked lists as in bucket sorting.

As a theoretical exercise, in Section 5, we also consider the case when this sorting has to be done stably. We present an algorithm with identical performance that is also stable. Similar to the other in-place stable sorting algorithms e.g., comparison-based sorting [10], this algorithm is quite detailed and needs very careful management of keys as they are permuted. The resulting algorithm is likely not of practical value, but it is still fundamentally important to know that bucket and radix sorting can indeed be solved stably in $O(n)$ time with only $O(1)$ words of extra space. For example, even though comparison-based sorting has been well studied at least since 60's, it was not until much later that optimal, stable in-place comparison-based sorting was developed [10].

Arbitrary word length. Another question of fundamental theoretical interest is whether the recently discovered integer sorting algorithms that work with long keys and sort in $o(n \log n)$ time, such as [3, 1, 5], need any auxiliary space. In Section 4, we present a black-box transformation from any RAM sorting algorithm to an sorting algorithm which uses only $O(1)$ extra space, and retains the same time bounds. As a result, the running time bounds of [1, 5] can now be matched with only $O(1)$ extra space. This transformation relies on a fairly natural technique of compressing a portion of the input to make space for simulating space-inefficient RAM sorting algorithms.

Definitions. Formally, we are given a sequence S of n elements. The problem is to sort S according to the integer keys, under the following assumptions:

- (i) Each element has an integer key within the interval $[1, U]$.
- (ii) The following unit-cost operations are allowed on S : (a) indirect address of any position of S ; (b) read-only access to the key of any element; (c) exchange of the positions of any two elements.
- (iii) The following unit-cost operations are allowed on integer values of $O(\log U)$ bits: addition, subtraction, bitwise AND/OR and unrestricted bit shift.
- (iv) Only $O(1)$ auxiliary words of memory are allowed; each word had $\log U$ bits.

For the sake of presentation, we will refer to the elements' keys as if they were the input elements. For example, for any two elements x, y , instead of writing that the key of x is less than the key of y we will simply write $x < y$. We also need a precise definition of the *rank* of an element in a sequence when multiple occurrences of keys are allowed: the *rank* of an element x_i in a sequence $x_1 \dots x_t$ is the cardinality of the multiset $\{x_j \mid x_j < x_i \text{ or } (x_j = x_i \text{ and } j \leq i)\}$.

2 Stable Sorting for Modifiable Keys

We now describe our simple algorithm for (stable) radix sort without additional memory.

Gaining space. The first observation is that numbers in sorted order have less entropy than in arbitrary order. In particular, n numbers from a universe of u have binary entropy $n \log u$ when the order is unspecified, but only $\log \binom{u}{n} = n \log u - \Theta(n \log n)$ in sorted order. This suggests that we can “compress” sorted numbers to gain more space:

Lemma 1. *A list of n integers in sorted order can be represented as: (a) an array $A[1 \dots n]$ with the integers in order; (b) an array of n integers, such that the last $\Theta(n \log n)$ bits of the array are zero. Furthermore, there exist in-place $O(n)$ time algorithms for switching between representations (a) and (b).*

Proof. One can imagine many representations (b) for which the lemma is true. We note nonetheless that some care is needed, as some obvious representations will in fact not lead to in-place encoding. Take for instance the appealing approach of replacing $A[i]$ by $A[i] - S[i - 1]$, which makes numbers tend to be small (the average value is $\frac{u}{n}$). Then, one can try to encode the difference using a code optimized for smaller integers, for example one that represents a value x using $\log x + O(\log \log x)$ bits. However, the obvious encoding algorithm will not be in-place: even though the scheme is guaranteed to save space over the entire array, it is possible for many large values to cluster at the beginning, leading to a rather large prefix being in fact expanded. This makes it hard to construct the encoding in the same space as the original numbers, since we need to shift a lot of data to the right before we start seeing a space saving.

As it will turn out, the practical performance of our radix sort is rather insensitive to the exact space saving achieved here. Thus, we aim for a representation which makes in-place encoding particularly easy to implement, sacrificing constant factors in the space saving.

First consider the most significant bit of all integers. Observe that if we only remember the minimum i such that $A[i] \geq u/2$, we know all most significant bits (they are zero up to i and one after that). We will encode the last $n/3$ values in the array more compactly, and use the most significant bits of $A[1 \dots \frac{2}{3}n]$ to store a stream of $\frac{2}{3}n$ bits needed by the encoding.

We now break a number x into $\text{hi}(x)$, containing the upper $\lfloor \log_2(n/3) \rfloor$ bits, and $\text{lo}(x)$, with the low $\log u - \lfloor \log_2(n/3) \rfloor$ bits. For all values in $A[\frac{2}{3}n + 1 \dots n]$, we can throw away $\text{hi}(A[i])$ as follows. First we add $\text{hi}(A[\frac{2}{3}n + 1])$ zeros to the bit stream, followed by a one; then for every $i = \frac{2}{3}n + 2, \dots, n$ we add $\text{hi}(A[i]) - \text{hi}(A[i - 1])$ zeros, followed by a one. In total, the stream contains exactly $n/3$ ones (one per element), and exactly $\text{hi}(A[n]) \leq n/3$ zeros. Now we simply compact $\text{lo}(A[\frac{2}{3}n + 1]), \dots, \text{lo}(A[n])$ in one pass, gaining $\frac{n}{3} \lfloor \log_2(n/3) \rfloor$ free bits. \square

An unstable algorithm. Even just this compression observation is enough to give a simple algorithm, whose only disadvantage is that it is unstable. The algorithm has the following structure:

1. sort the subsequence $S[1 \dots (n/\log n)]$ using the optimal in-place mergesort in [10].
2. compress $S[1 \dots (n/\log n)]$ by Lemma 1, generating $\Omega(n)$ bits of free space.
3. radix sort $S[(n/\log n) + 1 \dots n]$ using the free space.
4. uncompress $S[1 \dots (n/\log n)]$.
5. merge the two sorted sequences $S[1 \dots (n/\log n)]$ and $S[(n/\log n) + 1 \dots n]$ by using the in-place, linear time merge in [10].

The only problematic step is 3. The implementation of this step is based on the cycle leader permuting approach where a sequence A is re-arranged by following the cycles of a permutation π . First $A[1]$ is sent in its final position $\pi(1)$. Then, the element that was in $\pi(1)$ is sent to its final position $\pi(\pi(1))$. The process proceeds in this way until the cycle is closed, that is until the element that is moved in position 1 is found. At this point, the elements starting from $A[2]$ are scanned until a new cycle leader $A[i]$ (i.e. its cycle has not been walked through) is found, $A[i]$'s cycle is followed in its turn, and so forth.

To sort, we use $2n^\epsilon$ counters $c_1, \dots, c_{n^\epsilon}$ and $d_1, \dots, d_{n^\epsilon}$. They are stored in the auxiliary words obtained in step 2. Each d_j is initialized to 0. With a first scan of the elements, we store in any c_i the number of occurrences of key i . Then, for each $i = 2 \dots n^\epsilon$, we set $c_i = c_{i-1} + 1$ and finally we set $c_1 = 1$ (in the end, for any i we have that $c_i = \sum_{j < i} c_j + 1$). Now we have all the information for the cycle leader process. Letting $j = (n/\log n) + 1$, we proceed as follows:

- (i) let i be the key of $S[j]$;
- (ii) if $c_i \leq j < c_{i+1}$ then $S[j]$ is already in its final position, hence we increment j by 1 and go to step (i);
- (iii) otherwise, we exchange $S[j]$ with $S[c_i + d_i]$, we increment d_i by 1 and we go to step (i).

Note that this algorithm is inherently unstable, because we cannot differentiate elements which should fall between c_i and $c_{i+1} - 1$, given the free space we have.

Stability through recursion. To achieve stability, we need more than n free bits, which we can achieve by bootstrapping with our own sorting algorithm, instead of merge sort. There is also an important practical advantage to the new stable approach: the elements are permuted much more conservatively, resulting in better cache performance.

1. recursively sort a constant fraction of the array, say $S[1 \dots n/2]$.
2. compress $S[1 \dots n/2]$ by Lemma 1, generating $\Omega(n \log n)$ bits of free space.
3. for a small enough constant γ , break the remaining $n/2$ elements into chunks of γn numbers. Each chunk is sorted by a classic radix sort algorithm which uses the available space.
4. uncompress $S[1 \dots n/2]$.
5. we now have $1 + 1/\gamma = O(1)$ sorted subarrays. We merge them in linear time using the stable in-place algorithm of [10].

We note that the recursion can in fact be implemented bottom up, so there is no need for a stack of superconstant space. For the base case, we can use bubble sort when we are down to $n \leq \sqrt{n_0}$ elements, where n_0 is the original size of the array at the top level of the recursion.

Steps 2 and 4 are known to take $O(n)$ time. For step 3, note that radix sort in base R applied to N numbers requires $N + R$ additional words of space, and takes time $O(N \log_R u)$. Since we have a free space of $\Omega(n \log n)$ bits or $\Omega(n)$ words, we can set $N = R = \gamma n$, for a small enough constant γ . As we always have $n = \Omega(\sqrt{n_0}) = u^{\Omega(1)}$, radix sort will take linear time.

The running time is described by the recursion $T(n) = T(n/2) + O(n)$, yielding $T(n) = O(n)$.

A self-contained algorithm. Unfortunately, all algorithms so far use in-place stable merging algorithm as in [10]. We want to remove this dependence, and obtain a simple and practical sorting algorithm. By creating free space through compression at the right times, we can instead use a simple merging implementation that needs additional space. We first observe the following:

Lemma 2. *Let $k \geq 2$ and $\alpha > 0$ be arbitrary constants. Given k sorted lists of n/k elements, and αn words of free space, we can merge the lists in $O(n)$ time.*

Proof. We divide space into blocks of $\alpha n/(k+1)$ words. Initially, we have $k+1$ free blocks. We start merging the lists, writing the output in these blocks. Whenever we are out of free blocks, we look for additional blocks which have become free in the original sorted lists. In each list, the merging pointer may be inside some block, making it yet unavailable. However, we can only have k such partially consumed blocks, accounting for less than $k \frac{\alpha n}{k+1}$ wasted words of space. Since in total there are αn free words, there must always be at least one block which is available, and we can direct further output into it.

At the end, we have the merging of the lists, but the output appears in a nontrivial order of the blocks. Since there are $(k+1)(1+1/\alpha) = O(1)$ blocks in total, we can remember this order using constant additional space. Then, we can permute the blocks in linear time, obtaining the true sorted order. \square

Since we need additional space for merging, we can never work with the entire array at the same time. However, we can now use a classic sorting idea, which is often used in introductory algorithms courses to illustrate recursion (see, e.g. [2]). To sort n numbers, one can first sort the first $\frac{2}{3}n$ numbers (recursively), then the last $\frac{2}{3}n$ numbers, and then the first $\frac{2}{3}n$ numbers again. Though normally this algorithm gives a running time of $\omega(n^2)$, it works efficiently in our case because we do not need recursion:

1. sort $S[1 \dots n/3]$ recursively.
2. compress $S[1 \dots \frac{n}{3}]$, and sort $S[\frac{n}{3}+1 \dots n]$ as before: first radix sort chunks of γn numbers, and then merge all chunks by Lemma 2 using the available space. Finally, uncompress $S[1 \dots \frac{n}{3}]$.
3. compress $S[\frac{2n}{3}+1 \dots n]$, which is now sorted. Using Lemma 2, merge $S[1 \dots \frac{n}{3}]$ with $S[\frac{n}{3}+1 \dots \frac{2n}{3}]$. Finally uncompress.
4. once again, compress $S[1 \dots \frac{n}{3}]$, merge $S[\frac{n}{3}+1 \dots \frac{2n}{3}]$ with $S[\frac{2n}{3}+1 \dots n]$, and uncompress.

Note that steps 2–4 are linear time. Then, we have the recursion $T(n) = T(n/3) + O(n)$, solving to $T(n) = O(n)$. Finally, we note that stability of the algorithm follows immediately from stability of classic radix sort and stability of merging.

Practical experience. The algorithm is surprisingly effective in practice. It can be implemented in about 150 lines of C code. Experiments with sorting 1-10 million 32-bit numbers on a Pentium machine indicate the algorithm is roughly 2.5 times slower than radix sort with additional memory, and slightly faster than quicksort (which is not even stable).

3 Unstable Sorting for Read-only Keys

3.1 Simulating auxiliary bits

With the bit stealing technique [9], a bit of information is encoded in the relative order of a pair of elements with different keys: the pair is maintained in increasing order to encode a 0 and vice

versa. The obvious drawback of this technique is that the cost of accessing a word of w encoded bits is $O(w)$ in the worst case (no word-level parallelism). However, if we modify an encoded word with a series of l increments (or decrements) by 1, the total cost of the entire series is $O(l)$ (see [2]).

To find pairs of distinct elements, we go from S to a sequence $Z'Y'XY''Z''$ with two properties. (i) For any $z' \in Z'$, $y' \in Y'$, $x \in X$, $y'' \in Y''$ and $z'' \in Z''$ we have that $z' < y' < x < y'' < z''$. (ii) Let $m = \alpha \lceil n/\log n \rceil$, for a suitable constant α . Y' is composed by the element y'_m with rank m plus all the other elements equal to y'_m . Y'' is composed by the element y''_m with rank $n-m+1$ plus all the other elements equal to y''_m . To obtain the new sequence we use the in-place, linear time selection and partitioning algorithms in [6, 7]. If X is empty, the task left is to sort Z' and Z'' , which can be accomplished with any optimal, in-place mergesort (e.g. [10]). Let us denote $Z'Y'$ with M' and $Y''Z''$ with M'' . The m pairs of distinct elements $(M'[1], M''[1]), (M'[2], M''[2]), \dots, (M'[m], M''[m])$ will be used to encode information.

Since the choice of the constant α does not affect the asymptotic complexity of the algorithm, we have reduced our problem to a problem in which we are allowed to use a special *bit memory* with $O(n/\log n)$ bits where each bit can be accessed and modified in constant time but without word-level parallelism.

3.2 Simulating auxiliary memory for permuting

With the internal buffering technique [8], some of the elements are used as placeholders in order to simulate a working area and permute the other elements at lower cost. In our unstable sorting algorithm we use the basic idea of internal buffering in the following way. Using the selection and partitioning algorithms in [6, 7], we pass from the original sequence S to ABC with two properties. (i) For any $a \in A$, $b \in B$ and $c \in C$, we have that $a < b < c$. (ii) B is composed of the element b' with rank $\lceil n/2 \rceil$ plus all the other elements equal to b' . We can use BC as an auxiliary memory in the following way. The element in the first position of BC is the *separator element* and will not be moved. The elements in the other positions of BC are placeholders and will be exchanged with (instead of being overwritten by) elements from A in any way the computation on A (in our case the sorting of A) may require. The “emptiness” of any location i of the simulated working area in BC can be tested in $O(1)$ time by comparing the separator element $BC[1]$ with $BC[i]$: if $BC[1] \leq BC[i]$ the i th location is “empty” (that is, it contains a placeholder), otherwise it contains one of the elements in A .

Let us suppose we can sort the elements in A in $O(|A|)$ time using BC as working area. After A is sorted we use the partitioning algorithm in [6] to separate the elements equal to the separator element ($BC[1]$) from the elements greater than it (the computation on A may have altered the original order in BC). Then we just re-apply the same process to C , that is we divide it into $A'B'C'$, we sort A' using $B'C'$ as working area and so forth. Clearly, this process requires $O(n)$ time and when it terminates the elements are sorted. Obviously, we can divide A into $p = O(1)$ equally sized subsequences $A_1, A_2 \dots A_p$, then sort each one of them using BC as working area and finally fuse them using the in-place, linear time merging algorithm in [10]. Since the choice of the constant p does not affect the asymptotic complexity of the whole process, we have reduced our problem to a new problem, in which we are allowed to use a special *exchange* memory of $O(n)$ locations, where each location can contain *input elements only* (no integers or any other kind of data). Any element can be moved to and from any location of the exchange memory in $O(1)$ time.

3.3 The reduced problem

By blending together the basic techniques seen above, we can focus on a reduced problem in which assumption (iv) is replaced by:

(iv) Only $O(1)$ words of normal auxiliary memory and two kinds of special auxiliary memory are allowed:

- (a) A random access bit memory \mathcal{B} with $O(n/\log n)$ bits, where each bit can be accessed in $O(1)$ time (no word-level parallelism).
- (b) A random access exchange memory \mathcal{E} with $O(n)$ locations, where each location can contain only elements from S and they can be moved to and from any location of \mathcal{E} in $O(1)$ time.

If we can solve the reduced problem in $O(n)$ time we can also solve the original problem with the same asymptotic complexity. However, the resulting algorithm will be unstable because of the use of the internal buffering technique with a large pool of placeholder elements.

3.4 The naive approach

Despite the two special auxiliary memories, solving the reduced problem is not easy. Let us consider the following naive approach. We proceed as in the normal bucket sorting: one bucket for each one of the n^ϵ range values. Each bucket is a linked list: the input elements of each bucket are maintained in \mathcal{E} while its auxiliary data (e.g. the pointers of the list) are maintained in \mathcal{B} . In order to amortize the cost of updating the auxiliary data (each pointer requires a word of $\Theta(\log n)$ bits and \mathcal{B} does not have word-level parallelism), each bucket is a linked list of *slabs* of $\Theta(\log^2 n)$ elements each (\mathcal{B} has only $O(n/\log n)$ bits). At any time each bucket has a partially full *head slab* which is where any new element of the bucket is stored. Hence, for each bucket we need to store in \mathcal{B} a word of $O(\log \log n)$ bits with the position in the head slab of the last element added. The algorithm proceeds as usual: each element in S is sent to its bucket in $O(1)$ time and is inserted in the bucket's head slab. With no word-level parallelism in \mathcal{B} the insertion in the head slab requires $O(\log \log n)$ time. Therefore, we have an $O(n \log \log n)$ time solution for the reduced problem and, consequently, an unstable $O(n \log \log n)$ time solution for the original problem.

This simple strategy can be improved by dividing the head slab of a bucket into *second level slabs* of $\Theta(\log \log n)$ elements each. As for the first level slabs, there is a partially full, second level head slab. For any bucket we maintain two words in \mathcal{B} : the first one has $O(\log \log \log n)$ bits and stores the position of the last element inserted in the second level head slab; the second one has $O(\log \log n)$ bits and stores the position of the last full slab of second level contained in the first level head slab. Clearly, this gives us an $O(n \log \log \log n)$ time solution for the reduced problem and the corresponding unstable solution for the original problem. By generalizing this approach to the extreme, we end up with $O(\log^* n)$ levels of slabs, an $O(n \log^* n)$ time solution for the reduced problem and the related unstable solution for the original problem.

3.5 The pseudo pointers

Unlike bit stealing and internal buffering which were known earlier, the pseudo pointers technique has been specifically designed for improving the space complexity in integer sorting problems. Basically, in this technique a set of elements with distinct keys is used as a pool of pre-set, read-only pointers in order to simulate efficiently traversable and updatable linked lists. Let us show how to use this basic idea in a particular procedure that will be at the core of our optimal solution for the reduced problem.

Let d be the number of distinct keys in S . We are given two sets of d input elements with distinct keys: the sets \mathcal{G} and \mathcal{P} of *guides* and *pseudo pointers*, respectively. The guides are given us *in sorted order* while the pseudo pointers form a sequence in arbitrary order. Finally, we are given a multiset \mathcal{I} of d input elements (i.e. two elements of \mathcal{I} can have equal keys). The procedure

uses the guides, the pseudo pointers and the exchange memory to sort the d input elements of \mathcal{I} in $O(d)$ time.

We use three groups of contiguous locations in the exchange memory \mathcal{E} . The first group H has n^ϵ locations (one for each possible value of the keys). The second group L has n^ϵ slots of two adjacent locations each. The last group R has d locations, the elements of \mathcal{I} will end up here in sorted order. H , L and R are initially empty. We have two main steps.

First. For each $s \in \mathcal{I}$, we proceed as follows. Let p be the leftmost pseudo pointer still in \mathcal{P} . If the s th location of H is empty, we move p from \mathcal{P} to $H[s]$ and then we move s from \mathcal{I} to the first location of $L[p]$ (i.e. the first location of the p th slot of L) leaving the second location of $L[p]$ empty. Otherwise, if $H[s]$ contains an element p' (a pseudo pointer) we move s from \mathcal{I} to the first location of $L[p]$, then we move p' from $H[s]$ to the second location of $L[p]$ and finally we move p from \mathcal{P} to $H[s]$.

Second. We scan the guides in \mathcal{G} from the smallest to the largest one. For a guide $g \in \mathcal{G}$ we proceed as follows. If the g th location of H is empty then there does not exist any element equal to g among the ones to be sorted (and initially in \mathcal{I}) and hence we move to the next guide. Otherwise, if $H[G]$ contains a pseudo pointer p , there is at least one element equal to g among the ones to be sorted and this element is currently stored in the first location of the p th slot of L . Hence, we move that element from the first location of $L[p]$ to the leftmost empty location of R . After that, if the second location of $L[p]$ contains a pseudo pointer p' , there is another element equal to g and we proceed in the same fashion. Otherwise, if the second location of $L[p]$ is empty then there are no more elements equal to g among the ones to be sorted and therefore we can focus on the next guide element.

Basically, the procedure is bucket sorting where the auxiliary data of the list associated to each bucket (i.e. the links among elements in the list) *is implemented by pseudo pointers in \mathcal{P}* instead of storing it explicitly in the bit memory (which lacks of word-level parallelism and is inefficient in access). It is worth noting that the buckets' lists implemented with pseudo pointers are spread over an area that is larger than the one we would obtain with explicit pointers (that is because each pseudo pointer has a key of $\log n^\epsilon$ bits while an explicit pointer would have only $\log d$ bits).

3.6 The optimal solution

We can now describe the algorithm, which has three main steps.

First. Let us assume that for any element $s \in S$ there is at least another element with the same key. (Otherwise, we can easily reduce to this case in linear time: we isolate the $O(n^\epsilon)$ elements that do not respect the property, we sort them with the in-place mergesort in [10] and finally we merge them after the other $O(n)$ elements are sorted.) With this assumption, we extract from S two sets \mathcal{G} and \mathcal{P} of d input elements with distinct keys (this can be easily achieved in $O(n)$ time using only the exchange memory \mathcal{E}). Finally we sort \mathcal{G} with the optimal in-place mergesort in [10].

Second. Let S' be the sequence with the $(O(n))$ input elements left after the first step. Using the procedure in § 3.5 (clearly, the elements in the sets \mathcal{G} and \mathcal{P} computed in the first step will be the guides and pseudo pointers used in the procedure), we sort each block B_i of S' with d contiguous elements. After that, let us focus on the first $t = \Theta(\log \log n)$ consecutive blocks B_1, B_2, \dots, B_t . We distribute the elements of these blocks into $\leq t$ groups G_1, G_2, \dots in the following way. Each group G_j can contain between d and $2d$ elements and is allocated in the exchange memory \mathcal{E} . The largest element in a group is its *pivot*. The number of elements in a group is stored in a word of $\Theta(\log d)$ bits allocated in the bit memory \mathcal{B} . Initially there is only one group and is empty. In the i th step of the distribution we scan the elements of the i th block B_i . As long as the elements of B_i are less than or equal to the pivot of the first group we move them into it. If, during the process, the group becomes full, we select its median element and partition the group into two new groups

(using the selection and partitioning algorithms in [6, 7]). When, during the scan, the elements of B_i become greater than the pivot of the first group, we move to the second group and continue in the same fashion. It is important to notice that the number of elements in a group (stored in a word of $\Theta(\log d)$ bits in the bit memory \mathcal{B}) is updated by increments by 1 (and hence the total cost of updating the number of elements in any group is linear in the final number of elements in that group, see [2]). Finally, when all the elements of the first $t = \Theta(\log \log n)$ consecutive blocks B_1, B_2, \dots, B_t have been distributed into groups, we sort each group using the procedure in § 3.5 (when a group has more than d elements, we sort them in two batches and then merge them with the in-place, linear time merging in [10]). The whole process is repeated for the second $t = \Theta(\log \log n)$ consecutive blocks, and so forth.

Third. After the second step, the sequence S' (which contains all the elements of S with the exclusion of the guides and pseudo pointers, see the first step) is composed by contiguous subsequences S'_1, S'_2, \dots which are *sorted* and contain $\Theta(d \log \log n)$ elements each (where d is the number of distinct elements in S). Hence, if we see S' as composed by contiguous runs of elements with the same key, we can conclude that the number of runs of S' is $O(n / \log \log n)$. Therefore S' can be sorted in $O(n)$ time using the naive approach described in § 3.4 with only the following simple modification. As long as we are inserting the elements of a single run in a bucket, we maintain the position of the last element inserted in the head slab of the bucket in a word of auxiliary memory (we can use $O(1)$ of them) instead of accessing the inefficient bit memory \mathcal{B} at any single insertion. When the current run is finally exhausted, we copy the position in the bit memory. Finally, we sort \mathcal{P} and we merge \mathcal{P} , \mathcal{A} and S' (once again, using the sorting and merging algorithms in [10]).

3.7 Discussion: Stability and Read-only Keys

Let us focus on the reasons why the algorithm of this section is not stable. The major cause of instability is the use of the basic internal buffering technique in conjunction with large ($\omega(\text{polylog}(n))$) pools of placeholder elements. This is clearly visible even in the first iteration of the process in § 3.2: after being used to permute A into sorted order, the placeholder elements in BC are left permuted in a completely arbitrary way and their initial order is lost.

4 Reducing Space in any RAM Sorting Algorithm

In this section, we consider the case of sorting integers of $w = \omega(\log n)$ bits. We show a black box transformation from any sorting algorithm on the RAM to a stable sorting algorithm with the same time bounds which only uses $O(1)$ words of additional space. Our reduction needs to modify keys. Furthermore, it requires randomization for large values of w .

We first remark that an algorithm that runs in time $t(n)$ can only use $O(t(n))$ words of space in most realistic models of computation. In models where the algorithm is allowed to write $t(n)$ arbitrary words in a larger memory space, the space can also be reduced to $O(t(n))$ by introducing randomization, and storing the memory cells in a hash table.

Small word size. We first deal with the case $w = \text{polylog}(n)$. The algorithm has the following structure:

1. sort $S[1 \dots n / \log n]$ using in-place stable merge sort [10]. Compress these elements by Lemma 1 gaining $\Omega(n)$ bits of space.

2. since $t(n) = O(n \log n)$, the RAM sorting algorithm uses at most $O(t(n) \cdot w) = O(n \text{polylog}(n))$ bits of space. Then we can break the array into chunks of $n/\log^c n$ elements, and sort each one using the available space.
3. merge the $\log^c n$ sorted subarrays.
4. uncompress $S[1 \dots n/\log n]$ and merge with the rest of the array by stable in-place merging [10].

Steps 1 and 4 take linear time. Step 2 requires $\log^c n \cdot t(n/\log^c n) = O(t(n))$ because $t(n)$ is convex and bounded in $[n, n \log n]$. We note that step 2 can always be made stable, since we can afford a label of $O(\log n)$ bits per value.

It remains to show that step 3 can be implemented in $O(n)$ time. In fact, this is a combination of the merging technique from Lemma 2 with an atomic heap [4]. The atomic heap can maintain a priority queue over $\text{polylog}(n)$ elements with constant time per insert and extract-min. Thus, we can merge $\log^c n$ lists with constant time per element. The atomic heap can be made stable by adding a label of $c \log \log n$ bits for each element in the heap, which we have space for. The merging of Lemma 2 requires that we keep track of $O(k/\alpha)$ subarrays, where $k = \log^c n$ was the number of lists and $\alpha = 1/\text{polylog}(n)$ is fraction of additional space we have available. Fortunately, this is only $\text{polylog}(n)$ values to record, which we can afford.

Large word size. For word size $w \geq \log^{1+\epsilon} n$, the randomized algorithm of [1] can sort in $O(n)$ time. Since this is the best bound one can hope for, it suffices to make this particular algorithm in-place, rather than give a black-box transformation. We use the same algorithm from above. The only challenge is to make step 2 work: sort n keys with $O(n \text{polylog}(n))$ space, even if the keys have $w > \text{polylog}(n)$ bits.

We may assume $w \geq \log^3 n$, which simplifies the algorithm of [1] to two stages. In the first stage, a signature of $O(\log^2 n)$ bits is generated for each input value (through hashing), and these signatures are sorted in linear time. Since we are working with $O(\log^2 n)$ -bit keys regardless of the original w , this part needs $O(n \text{polylog}(n))$ bits of space, and it can be handled as above.

From the sorted signatures, an additional pass extracts a subkey of $w/\log n$ bits from each input value. Then, these subkeys are sorted in linear time. Finally, the order of the original keys is determined from the sorted subkeys and the sorted signatures.

To reduce the space in this stage, we first note that the algorithm for extracting subkeys does not require additional space. We can then isolate the subkey from the rest of the key, using shifts, and group subkeys and the remainder of each key in separate arrays, taking linear time. This way, by extracting the subkeys instead of copying them we require no extra space. We now note that the algorithm in [1] for sorting the subkeys also does not require additional space. At the end, we recompose the keys by applying the inverse permutation to the subkeys, and shifting them back into the keys.

Finally, sorting the original keys only requires knowledge of the signatures and *order* information about the subkeys. Thus, it requires $O(n \text{polylog}(n))$ bits of space, which we have. At the end, we find the sorted order of the original keys and we can implement the permutation in linear time.

5 Stable sorting for read-only keys

In the following we will denote n^ϵ with r . Before we begin, let us recall that two consecutive sequences X and Y , possibly of different sizes, can be exchanged stably, in-place and in linear time with three sequence reversals, since $YX = (X^R Y^R)^R$. Let us give a short overview of the algorithm. We have three phases.

Preliminary phase (§ 5.1). The purpose of this phase is to obtain some collections of elements to be used with the three techniques described in § 3. We extract $\Theta(n/\log n)$ smallest and largest elements of S . They will form an encoded memory of $\Theta(n/\log n)$ bits. Then, we extract from the remaining sequence $\Theta(n^\epsilon)$ smallest elements and divide them into $O(1)$ *jump zones* of equal length. After that, we extract from the remaining sequence some equally sized sets of distinct elements. Each set is collected into a contiguous zone. At the end of the phase, we have *guide*, *distribution*, *pointer*, *head* and *spare* zones.

Aggregating phase (§ 5.2). After the preliminary phase, we have reduced the problem to sorting a smaller sequence S' (still of $O(n)$ size) using various sequences built to be used with the basic techniques in § 3. Let d be the number of distinct elements in S' (computed during the preliminary phase). The objective of this phase is to sort each subsequence of size $\Theta(d \text{polylog}(n))$ of the main sequence S' . For any such subsequence S'_l , we first find a set of pivots and then sort S'_l with a distributive approach. The guide zone is sorted and is used to retrieve in sorted order lists of equal elements produced by the distribution. The distribution zone provides sets of pivots elements that are progressively moved into one of the spare zones. The head zone furnishes placeholder elements for the distributive processes. The distributive process depends on the hypothesis that each d contiguous elements of S'_l are sorted. The algorithm for sorting $\Theta(d)$ contiguous elements stably, in $O(d)$ time and $O(1)$ space (see § 5.2.1) employs the pseudo pointers technique and the guide, jump, pointer and spare zones are crucial in this process.

Final phase (§ 5.3). After the aggregating phase the main sequence S' has all its subsequences of $\Theta(d \text{polylog}(n))$ elements in sorted order. With an iterative merging process, we obtain from S' two new sequences: a small sequence containing $O(d \log^2 n)$ sorted elements; a large sequence still containing $O(n)$ elements but with an important property: the length of any subsequence of equal elements is multiple of a suitable number $\Theta(\log^2 n)$. By exploiting its property, the large sequence is sorted using the encoded memory and merged with the small one. Finally, we take care of all the zones built in the preliminary phase. Since they have sizes either $O(n/\log n)$ or $O(n^\epsilon)$, they can be easily sorted within our target bounds.

5.1 Preliminary Phase

The preliminary phase has two main steps described in Sections 5.1.1 and 5.1.2.

5.1.1 Encoded memory

We start by collecting some pairs of distinct elements. We go from S to a sequence $Z'Y'XY''Z''$ with the same two properties we saw in § 3.1. We use the linear time selection and partitioning in [6, 7] which are also stable. Let us maintain the same notations used in § 3.1. We end up with m pairs of distinct elements $(M'[1], M''[1]), \dots, (M'[m], M''[m])$ to encode information by bit stealing. We use the encoded memory based on these pairs as if it were actual memory (that is, we will allocate arrays, we will index and modify entries of these arrays, etc). However, in order not to lose track of the costs, the names of encoded structures will always be written in the following way: \mathbb{I} , \mathbb{U} , etc.

We allocate two arrays \mathbb{I}_{bg} and \mathbb{I}_{en} , each one with $r = n^\epsilon$ entries of 1 bit each. The entries of both arrays are set to 0. \mathbb{I}_{bg} and \mathbb{I}_{en} will be used each time the procedure in § 5.2.1 of the aggregating phase is invoked.

5.1.2 Jump, guide, distribution, pointer, head and spare zones

The second main step of this phase has six sub-steps.

First. Let us suppose $|X| > n/\log n$ (otherwise we sort it with the mergesort in [10]). Using the selection and partitioning in [6, 7], we go from X to JX' such that J is composed by the element j^* with rank $3r + 1 = 3n^\epsilon + 1$ (in X) plus all the elements (in X) $\leq j^*$. Then, we move the rightmost element equal to j^* in the last position of J (easily done in $O(|J|)$ and stably with a sequence exchange).

Second. Let us suppose $|X'| > n/\log n$. With this step and the next one we separate the elements which appear more than 7 times. Let us allocate in our encoded memory of $m = O(n/\log n)$ bits an encoded array \mathbb{I} with $r (= n^\epsilon)$ entries of 4 bits each. All the entries are initially set to 0. Then, we start scanning X' from left to right. For any element $u \in X'$ accessed during the scan, if $\mathbb{I}[u] \leq 7$, we increment $\mathbb{I}[u]$ by one.

Third. We scan X' again. Let $u \in X'$ be the i th element accessed. If $\mathbb{I}[u] < 7$, we decrement $\mathbb{I}[u]$ by 1 and exchange $X'[i] (= u)$ with $J[i]$. At the end of the scan we have that $J = WJ''$, where W contains the elements of X' occurring less than 7 times in X' . Then, we have to gather the elements previously in J and now scattered in X' . We accomplish this with the partitioning in [6], using $J[|J|]$ to discern between elements previously in J and the ones belonging to X' (we know that $J[|J|]$ is equal to j^* and, for any $j \in J$ and any $x' \in X'$, $j \leq J[|J|] < x'$). After that we have $WJ''J'X''$ where the elements in J' are the ones previously in J and exchanged during the scan of X' . We exchange W with J' ending up with JWX'' .

Fourth. We know that each element of X'' occurs at least 7 times in it. We also know that the entries of \mathbb{I} encode either 0 or 7. We scan X'' from left to right. Let $u \in X''$ be the i th element accessed. If $\mathbb{I}[u] = 7$, we decrement $\mathbb{I}[u]$ by one and we exchange $X''[i] (= u)$ with $J[i]$. After the scan we have that $J = GJ'''$, where, for any j , $G[j]$ was the leftmost occurrence of its kind in X'' (before the scan). Then, we sort G with the mergesort in [10] ($|G| = O(r) = O(n^\epsilon)$ and $\epsilon < 1$). After that, similarly to the third step, we gather the elements previously in J and now scattered in X'' because of the scan. We end up with the sequence $JWGJX'''$. We repeat the same process (only testing for $\mathbb{I}[u]$ equal to 6 instead of 7) to gather the leftmost occurrence of each distinct element in X''' into a zone D , ending up with the sequence $JWGDJX''''$.

Fifth. Each element of X'''' occurs at least 5 times in it and the entries of \mathbb{I} encode either 0 or 5. We scan X'''' , let $u \in X''''$ be the i th element accessed. If $\mathbb{I}[u] = 5$, we decrement $\mathbb{I}[u]$ by 1 and exchange $X''''[i] (= u)$ with $J[i]$. After the scan we have that $J = PJ'''$, where, for any j , $P[j]$ was the leftmost occurrence of its kind in X'''' (before the scan). Unlike the fourth step, we do not sort P . We repeat the process finding T_1, T_2, T_3 and H containing the second, third, fourth and fifth leftmost occurrence of each distinct element in X'''' , respectively. After any of these processes, we gather back the elements previously in J scattered in X'''' (same technique used in the third and fourth steps). We end up with the sequence $JWGDPT_1T_2T_3HS'$.

Sixth. Let us divide J into $J_1J_2J_3V$, where $|J_1| = |J_2| = |J_3| = r$ and $|V| = |J| - 3r$. We scan G , let $u \in G$ be the i th element accessed, we exchange $T_1[i]$, $T_2[i]$ and $T_3[i]$ with $J_1[u]$, $J_2[u]$ and $J_3[u]$, respectively.

5.1.3 Summing up

We will refer to J_1, J_2 and J_3 as *jump zones*. Zone G, D, P and H will be referred to as *guide, distribution, pointer* and *head zones*, respectively. Finally, T_1, T_2 and T_3 will be called *spare zones*. With the preliminary phase we have passed from the initial sequence S to $M'J_1J_2J_3VWGDPT_1T_2T_3HS'M''$. We allocated in the encoded memory two arrays \mathbb{I}_{bg} and \mathbb{I}_{en} . The encoded memory, \mathbb{I}_{bg} and \mathbb{I}_{en} , and the jump, guide, distribution, pointer, head and spare zones will be used in the next two phases to sort the S' . Zones V and W are a byproduct of the phase and will not have an active role

in the sorting of S' . The number of distinct elements in sequence S' is less than or equal to the sizes of guide, distribution, pointer and head zones. For the rest of the paper we will denote $|G|$ ($= |D| = |P| = |H|$) with d .

Lemma 3. *The preliminary phase requires $O(n)$ time, uses $O(1)$ auxiliary words and is stable.*

5.2 Aggregating Phase

Let us divide S' into k subsequences $S'_1 S'_2 \dots S'_k$ with $|S'_i| = d \log^\beta n$, for a suitable constant $\beta \geq 4$. Let $t = \log^\delta n$, for a suitable constant $\delta < 1$. We will assume that $d \geq (2t + 1) \log |S'_i|$. We leave the particular case where $d < (2t + 1) \log |S'_i|$ for the full paper. For a generic $1 \leq l \leq k$, let us assume that any $S'_{l'}$ with $l' < l$ has been sorted and that H is next to the left end of S'_l . To sort S'_l we have two main steps described in § 5.2.2 and § 5.2.3. They rely on the algorithm described in § 5.2.1.

5.2.1 Sorting $O(d)$ contiguous elements

Let us show how to exploit the two arrays \mathbb{I}_{bg} and \mathbb{I}_{en} , (in the encoded memory in the preliminary phase) and the jump, guide and pointer zones to sort a sequence A , with $|A| \leq d$, stably in $O(|A|)$ time and using $O(1)$ auxiliary words. The process has two steps.

First. We scan A , let $u \in A$ be the i th element accessed. Let $p = P[i]$ and $h = J_1[u]$. If $\mathbb{I}_{bg}[u] = 0$, we set both $\mathbb{I}_{bg}[u]$ and $\mathbb{I}_{en}[p]$ to 1. In any case, we exchange $J_1[u]$ ($= h$) with $J_2[p]$ and $A[i]$ ($= u$) with $J_3[p]$. Then, we exchange $P[i]$ ($= p$) with $J_1[u]$ (which is not h anymore).

Second. Let $j = |A|$. We scan G , let $g \in G$ be the i th element accessed. If $\mathbb{I}_{bg}[g] = 0$, we do nothing. Otherwise, let p be $J_1[g]$, we set $\mathbb{I}_{bg}[g] = 0$ and execute the following three steps. (i) We exchange $J_3[p]$ with $A[j]$, then $J_1[g]$ with $P[j]$ and finally $J_1[g]$ with $J_2[p]$. (ii) We decrease j by 1. (iii) If $\mathbb{I}_{en}[p] = 1$, we set $\mathbb{I}_{en}[p] = 0$ and the sub-process ends, otherwise, let p be $J_1[g + 1]$, and we go to (i).

Let us remark that the $O(|A|)$ entries of \mathbb{I}_{bg} and \mathbb{I}_{en} that are changed from 0 to 1 in the first step, are set back to 0 in the second one. \mathbb{I}_{bg} and \mathbb{I}_{en} have been initialized in the preliminary phase. We could not afford to re-initialize them every time we invoke the process (they have $r = n^\epsilon$ entries and $|A|$ may be $o(n^\epsilon)$).

Lemma 4. *Using the encoded arrays \mathbb{I}_{bg} , \mathbb{I}_{en} and the jump, guide and pointer zones, a sequence A with $|A| \leq d$ can be sorted stably, in $O(|A|)$ time and using $O(1)$ auxiliary words.*

5.2.2 Finding pivots for S'_l

We find a set of pivots $\{e_1, e_2, \dots, e_{p-1}, e_p\}$ with the following properties: (i) $|\{x \in S'_l \mid x < e_1\}| \leq d$; (ii) $|\{x \in S'_l \mid e_p < x\}| \leq d$; (iii) $|\{x \in S'_l \mid e_i < x < e_{i+1}\}| \leq d$, for any $1 \leq i < p$; (iv) $p = \Theta(\log^\beta n)$. In the end the pivots reside in the first p positions of D . We have four steps.

First. We allocate in the encoded memory an array \mathbb{P} with r entries of $\log |S'_l|$ bits, but we do not initialize each entry of \mathbb{P} . We initialize the only d of them we will need: for any $i = 1 \dots |G|$, we set $\mathbb{P}[G[i]]$ to 0.

Second. We scan S'_l from left to right. Let $u \in S'_l$ be the i th element accessed, we increment $\mathbb{P}[u]$ by 1.

Third. We sort the distribution zone D using the algorithm described in § 5.2.1.

Fourth. Let $i = 1$, $j = 0$ and $p = 0$. We repeat the following process until $i > |G|$. (i) Let $u = G[i]$, we set $j = j + \mathbb{P}[u]$. (ii) If $j < d$ we increase i by 1 and go to (i). If $j \geq d$, we increase p by 1, exchange $D[i]$ with $D[p]$, increase i by 1, set j to 0 and go to (i).

5.2.3 Sorting S'_l

Let p be the number of pivots for S'_l selected in § 5.2.2 and now residing in the first p positions of D . Let u be $\log |S'_l|$. Let us assume that H is next to the left end of S'_l . We have six steps.

First. Let $i = 0$ and $j = 0$. The following two steps are repeated until $i > p$: (i) we increase i by p/t and j by 1; (ii) we exchange $D[i]$ with $T_1[j]$. We will denote with p' the number of selected pivots, now temporarily residing in the first p' positions of T_1 .

Second. Let us divide S'_l into $q = |S'_l|/d$ blocks $B_1 B_2 \dots B_q$ of d elements each. We sort each B_i using the algorithm in § 5.2.1.

Third. With a sequence exchange we bring H next to the right end of S'_l . Let us divide H into $H_1 \hat{H}_1 \dots H_{p'} \hat{H}_{p'} H_{p'+1} H'$, where $|H_{p'+1}| = |H_i| = |\hat{H}_i| = u$. Let $f = |S'_l|/u$. We allocate the following arrays: (i) \mathbb{U}_{suc} and \mathbb{U}_{pre} both with $f + 2p' + 1$ entries of $\Theta(u)$ bits; (ii) \mathbb{H} and $\hat{\mathbb{H}}$ with $p' + 1$ and p' entries of $\Theta(\log u)$ bits; (iii) \mathbb{L} and $\hat{\mathbb{L}}$ with $p' + 1$ and p' entries of $\Theta(u)$ bits; (iv) \mathbb{N} and $\hat{\mathbb{N}}$ with $p' + 1$ and p' entries of $\Theta(u)$ bits. Each entry of any array is initialized to 0.

Fourth. In this step we want to transform S'_l and H in the following ways. We pass from S'_l to $U_1 U_2 \dots U_{f'-1} U_{f'} H''$, where the U_i 's are called *units*, for which the following holds.

- (i) $f' \geq f - (2p' + 1)$ and $|U_i| = u$, for any $1 \leq i \leq f'$.
- (ii) For any U_i , $1 \leq i \leq f'$, one of the following holds: (a) there exists a $1 \leq j \leq p'$ such that $x = T_1[j]$, for any $x \in U_i$; (b) $x < T_1[1]$, for any $x \in U_i$; (c) $T_1[p'] < x$, for any $x \in U_i$; (d) there exists a $1 \leq j' \leq p' - 1$ such that $T_1[j'] < x < T_1[j' + 1]$, for any $x \in U_i$.
- (iii) Let us call a *set of related units* a maximal set of units $\mathcal{U} = \{U_{i_1}, U_{i_2}, \dots, U_{i_z}\}$ for which one of the following conditions holds: (a) there exists a $1 \leq j \leq p'$ such that $x = T_1[j]$, for any $x \in U_i$ and for any $U_i \in \mathcal{U}$; (b) $x < T_1[1]$, for any $x \in U_i$ and for any $U_i \in \mathcal{U}$; (c) $T_1[p'] < x$, for any $x \in U_i$ and for any $U_i \in \mathcal{U}$; (d) there exists a $1 \leq j' \leq p' - 1$ such that $T_1[j'] < x < T_1[j' + 1]$, for any $x \in U_i$ and for any $U_i \in \mathcal{U}$. For any set of related units $\mathcal{U} = \{U_{i_1}, U_{i_2}, \dots, U_{i_z}\}$ we have that $\mathbb{U}_{suc}[i_y] = i_{y+1}$ and $\mathbb{U}_{pre}[i_{y+1}] = i_y$, for any $1 \leq y \leq z - 1$.

Concerning H'' and $H = H_1 \hat{H}_1 \dots H_{p'} \hat{H}_{p'} H_{p'+1} H'$. Before this step all the elements in H were the original ones gathered in § 5.1.2. After the fourth step, the following will hold.

- (iv) The elements in H' and H'' plus the elements in $H_i[\mathbb{H}[i] + 1 \dots u]$ and in $\hat{H}_{i'}[\hat{\mathbb{H}}[i'] + 1 \dots u]$, for any $1 \leq i \leq p' + 1$ and $1 \leq i' \leq p'$, form the original set of elements that were in H before the fourth step.
- (v) We have that: (a) $x < T_1[1]$, for any $x \in H_1[1 \dots \mathbb{H}[1]]$; (b) $x > T_1[p']$, for any $x \in H_{p'+1}[1 \dots \mathbb{H}[p' + 1]]$; (c) $T_1[i - 1] < x < T_1[i]$, for any $x \in H_i[1 \dots \mathbb{H}[i]]$ and any $2 \leq i \leq p'$; (d) $x = T_1[i]$, for any $x \in \hat{H}_i[1 \dots \hat{\mathbb{H}}[i]]$ and any $1 \leq i \leq p'$.
- (vi) (a) Let $j = \mathbb{L}[1]$ ($j = \mathbb{L}[p']$), U_j is the rightmost unit such that $x < T_1[1]$ ($x > T_1[p']$) for any $x \in U_j$. (b) For any $2 \leq i \leq p'$, let $j = \mathbb{L}[i]$, U_j is the rightmost unit such that $T_1[i - 1] < x < T_1[i]$ for any $x \in U_j$. (c) For any $2 \leq i \leq p'$, let $j = \hat{\mathbb{L}}[i]$, U_j is the rightmost unit such that $x = T_1[i]$ for any $x \in U_j$.
- (vii) (a) $\mathbb{N}[1]$ ($\mathbb{N}[p']$) is the number of $x \in S'_l$ such that $x < T_1[1]$ ($x > T_1[p']$). (b) For any $2 \leq i \leq p'$, $\mathbb{N}[i]$ is the number of $x \in S'_l$ such that $T_1[i - 1] < x < T_1[i]$. (c) For any $2 \leq i \leq p'$, $\hat{\mathbb{N}}[i]$ is the number of $x \in S'_l$ such that $x = T_1[i]$.

Let $h = \mathbb{H}[1]$, let $i = 1$ and let $j = 1$. We start the fourth step by scanning B_1 . If $B_1[i] < T_1[1]$, we increase h , $\mathbb{H}[1]$ and $\mathbb{N}[1]$ by 1, exchange $B[i]$ with $H_1[h]$ and increase i by 1. This sub-process goes on until one of the following two events happens: (a) h and $\mathbb{H}[1]$ are equal to $u + 1$; (b) $B_1[i] \geq T_1[1]$.

If event (a) happens, we exchange the u elements currently in H_1 with $S'_l[(j-1)u+1 \dots ju]$. Then, we set $\mathbb{U}_{pre}[j]$ to $\mathbb{L}[1]$, $\mathbb{U}_{suc}[\mathbb{L}[1]]$ to j and $\mathbb{L}[1]$ to j . After that, we set h and $\mathbb{H}[1]$ to 0 and we increment j by 1. Finally, we go back to the scanning of B_1 . Otherwise, if event (b) happens, we set h to $\hat{\mathbb{H}}[1]$ and we continue the scanning of B_1 but with the following sub-process: if $B_1[i] = T_1[1]$, we increase h , $\hat{\mathbb{H}}[1]$ and $\hat{\mathbb{N}}[1]$ by 1, exchange $B[i]$ with $\hat{H}_1[h]$ and increase i by 1. In its turn, this sub-process goes on until one of the following two events happens: (a') h and $\hat{\mathbb{H}}[1]$ are equal to $u+1$; (b') $B_1[i] > T_1[1]$. Similarly to what we did for event (a), if event (a') happens, we exchange the u elements currently in \hat{H}_1 with $S'_l[(j-1)u+1 \dots ju]$. Then, we set $\mathbb{U}_{pre}[j]$ to $\hat{\mathbb{L}}[1]$, $\mathbb{U}_{suc}[\hat{\mathbb{L}}[1]]$ to j and $\hat{\mathbb{L}}[1]$ to j . After that, we set h and $\hat{\mathbb{H}}[1]$ to 0 and we increment j by 1. Finally, we go back to the scanning of B_1 . Otherwise, if event (b') happens, we set h to $\mathbb{H}[2]$ and we continue the scanning of B_1 but with the following sub-process: if $B_1[i] < T_1[2]$, we increase h , $\mathbb{H}[2]$ and $\mathbb{N}[2]$ by 1, exchange $B[i]$ with $H_2[h]$ and increase i by 1. We continue in this fashion, possibly passing to $\mathbb{H}[2]$, $\mathbb{H}[3]$, etc, until B_1 is exhausted. Then, the whole process is applied to B_2 from the beginning. When B_2 is exhausted we pass to B_3 , B_4 and so forth until each block is exhausted.

Fifth. We start by exchanging H'' and $H_1\hat{H}_1 \dots H_{p'}\hat{H}_{p'}H_{p'+1}$. Let $h = 1$ and $h' = u - \mathbb{H}[1]$. We exchange the elements in $H_1[\mathbb{H}[1] + 1 \dots u]$ with the ones in $T_2[h \dots h']$. After that, we set $h = h'$, increment h' by $u - \hat{\mathbb{H}}[1]$ and exchange the elements in $\hat{H}_1[\hat{\mathbb{H}}[1] + 1 \dots u]$ with the ones in $T_2[h \dots h']$. We proceed in this fashion until $H_{p'+1}$ is done. Then we exchange the elements in $H''H'$ with the rightmost $|H''H'|$ ones in T_2 . After that we execute the following process to “link” the H_i ’s and \hat{H}_i ’s to their respective sets of related units. We start by setting $\mathbb{U}_{pre}[f' + 1]$ to $\mathbb{L}[1]$ and $\mathbb{U}_{suc}[\mathbb{L}[1]]$ to $f' + 1$. Then we set $\mathbb{U}_{pre}[f' + 2]$ to $\hat{\mathbb{L}}[1]$ and $\mathbb{U}_{suc}[\hat{\mathbb{L}}[1]]$ to $f' + 2$. We proceed in this fashion until we set $\mathbb{U}_{pre}[f' + 2p' + 1]$ to $\mathbb{L}[p' + 1]$ and $\mathbb{U}_{suc}[\mathbb{L}[p' + 1]]$ to $f' + 2p' + 1$. After that we execute the following process to bring each set of related units into a contiguous zone. Let $i = \lfloor \mathbb{N}[1]/u \rfloor + 1$, we exchange H_1 with U_i ; we swap the values in $\mathbb{U}_{pre}[i]$ and $\mathbb{U}_{pre}[f' + 1]$, then the values in $\mathbb{U}_{suc}[i]$ and $\mathbb{U}_{suc}[f' + 1]$; we set $\mathbb{U}_{pre}[\mathbb{U}_{suc}[f' + 1]] = f' + 1$ and $\mathbb{U}_{suc}[\mathbb{U}_{pre}[f' + 1]] = f' + 1$; finally, we set $\mathbb{U}_{suc}[\mathbb{U}_{pre}[i]] = i$. Then, let $j = \mathbb{U}_{pre}[i]$, we decrement i by 1, we exchange U_j with U_i ; we swap the values in $\mathbb{U}_{pre}[i]$ and $\mathbb{U}_{pre}[j]$, then the values in $\mathbb{U}_{suc}[i]$ and $\mathbb{U}_{suc}[j]$; we set $\mathbb{U}_{pre}[\mathbb{U}_{suc}[j]] = j$ and $\mathbb{U}_{suc}[\mathbb{U}_{pre}[j]] = j$; finally, we set $\mathbb{U}_{pre}[\mathbb{U}_{suc}[i]] = i$ and $\mathbb{U}_{suc}[\mathbb{U}_{pre}[i]] = i$. We proceed in this fashion until the entire set of related units of H_1 resides in $S'_l[1 \dots \mathbb{N}[1] - \mathbb{H}[1]]$ (H_1 now resides in $S'_l[\mathbb{N}[1] - \mathbb{H}[1] + 1 \dots \mathbb{N}[1] - \mathbb{H}[1] + u]$). After that, we apply the same process to \hat{H}_1 , H_2 , \hat{H}_2 and so forth until every set of related units has been compacted into a contiguous zone. We end up with the sequence $\overline{U}_1 H_1 \hat{U}_1 \hat{H}_1 \dots \overline{U}_{p'} H_{p'} \hat{U}_{p'} \hat{H}_{p'} \overline{U}_{p'+1} H_{p'+1} H'' H'$, where each \overline{U}_i contains the set of related units of H_i and each \hat{U}_i the set of related units of \hat{H}_i . Finally, we proceed to separate the elements of S'_l from the ones residing in $H_i[\mathbb{H}[i] + 1 \dots u]$ and $\hat{H}_{i'}[\mathbb{H}[i'] + 1 \dots u]$, for any $1 \leq i \leq p' + 1$ and any $1 \leq i' \leq p'$. Since the “intruders” were previously residing in T_2 , by the first and sixth steps in § 5.1.2 we know that any of them is less than any of the elements of S'_l . Therefore we can separate them with the stable partitioning algorithm in [6] and end up with the sequence $R_1 \hat{R}_1 \dots R_{p'} \hat{R}_{p'} R_{p'+1} H''' H'' H'$. Finally, we exchange the elements in the sequence $H = H''' H'' H'$ with the ones in T_2 , getting back in H its original (distinct) elements.

Sixth. After the fifth step we are left with the sequence $R_1 \hat{R}_1 \dots R_{p'} \hat{R}_{p'} R_{p'+1} H$ for which the following holds: (i) $\mathbb{N}[i] = |R_i|$ and $\hat{\mathbb{N}}[i'] = |\hat{R}_{i'}|$, for any $1 \leq i \leq p' + 1$ and any $1 \leq i' \leq p'$; (ii) for any $x \in R_1$, $x < T_1[1]$; (iii) for any $x \in R_{p'+1}$, $x < T_1[p']$; (iv) for any $x \in \hat{R}_i$, $x = T_1[i]$ ($1 \leq i \leq p'$); (v) for any $x \in R_i$, with $1 \leq i < p'$, $T_1[i] < x < T_1[i+1]$. We begin by moving H before R_1 with a sequence exchange. Since we do not need the p' pivots anymore, we put them back in their original positions in $D[1 \dots p]$ executing once again the process in the first step. Then, we allocate in the encoded memory an array \mathbb{R} with $p' + 1$ entries of $\Theta(u)$ bits. Let $i = 1$ and $\mathbb{R}[1] = 1$: for $j = 2, \dots, p' + 1$ we increment i by $\mathbb{N}[j-1] + \hat{\mathbb{N}}[j-1]$ and set $\mathbb{R}[j] = i$. After that, we execute a series of $p' + 1$ recursive invocations of the procedure here in § 5.2.3 (not the

whole sorting algorithm). We start by sorting $R_1 = S'_l[\mathbb{R}[1] \dots \mathbb{R}[2] - 1]$ recursively with the same procedure in this section: we use $S'_l[\mathbb{R}[1] \dots \mathbb{R}[2] - 1]$ in place of S'_l , $D[1 \dots p/t - 1]$ in place of D and so forth. After R_1 is sorted, we swap H and R_1 with a sequence exchange and proceed to sort R_2 : we use $S'_l[\mathbb{R}[2] \dots \mathbb{R}[3] - 1]$ in place of S'_l , $D[p/t + 1 \dots 2p/t - 1]$ in place of D and so forth. We proceed in this fashion until $R_{p'+1}$ is sorted and H is located after it again. We do not need anything particularly complex to handle the recursion with $O(1)$ words since there can be only $O(1)$ nested invocations.

Lemma 5. *The aggregation phase requires $O(n)$ time, uses $O(1)$ auxiliary words and is stable.*

5.3 Final Phase

The final phase has two main steps described in § 5.3.1 and § 5.3.2.

5.3.1 Sorting the main sequence S'

After the aggregating phase we are left with $S' = S'_1 S'_2 \dots S'_k$ where, for any $1 \leq i \leq k$, S'_i is sorted and $|S'_i| = d \log^\beta n$. Let $f = \log^2 n$. We have three steps.

First. We allocate in the encoded memory an array \mathbb{S}'_1 with $|S'_1|$ entries of one bit initially set to 1. Then, we scan S'_1 . During the scan, as soon as we encounter a subsequence $S'_1[i \dots i + f - 1]$ of equal elements (that is a subsequence of f consecutive equal elements) we set $\mathbb{S}'_1[i], \mathbb{S}'_1[i+1] \dots \mathbb{S}'_1[i + f - 2]$ and $\mathbb{S}'_1[i + f - 1]$ to 0. After that we use the partitioning algorithm in [6] to separate the elements of S'_1 with the corresponding entries in \mathbb{S}' set to 1 from the ones with their entries set to 0 (during the execution of the partitioning algorithm in [6], each time two elements are exchanged, the values of their entries in \mathbb{S}' are exchanged too). After the partitioning we have that $S'_1 = S''_1 O_1$ and the following conditions hold: (i) S''_1 and O_1 are still sorted (S'_1 was sorted and the partitioning is stable); (ii) the length of any maximal subsequence of consecutive equal elements of S''_1 is a multiple of f ; (iii) $|O_1| \leq df$. Then, we merge O_1 and S'_2 using the merging algorithm in [10], obtaining a sorted sequence S''_2 with $|S''_2| < 2d \log^\beta n$. After that, we apply to S''_2 the same process we applied to S'_1 ending up with $S''_2 = S''_2 O_2$ where conditions (i), (ii) and (iii) hold for S''_2 and O_2 too. We proceed in this fashion until each S'_k is done.

Second. We have that $S' = S'' O$. The following conditions hold: (i) O is sorted and $|O| \leq df$; (ii) the length of any maximal subsequence of consecutive equal elements of S'' is a multiple of f (and so $|S''|$ is a multiple of f too). Let $s = |S''|/f$ and let us divide S'' into s subsequences $F_1 F_2 \dots F_{s-1} F_s$ with $|F_i| = f$. We allocate in the encoded memory two arrays \mathbb{S}''_{pre} and \mathbb{S}''_{suc} , each one with s entries of $\Theta(\log n)$ bits. We also allocate an array \mathbb{C} with $r (= n^\epsilon)$ entries of $\Theta(\log n)$ bits, each one initialized to 0. Then, for each F_i from the rightmost to the leftmost one, we do the following: Let $v = F_i[1]$. If $\mathbb{C}[v] = 0$, we set $\mathbb{S}''_{suc}[i] = 0$ and $\mathbb{C}[v] = i$. Otherwise, if $\mathbb{C}[v] \neq 0$, we set $\mathbb{S}''_{suc}[i] = \mathbb{C}[v]$, $\mathbb{S}''_{pre}[\mathbb{C}[v]] = i$ and $\mathbb{C}[v] = i$.

Third. We scan \mathbb{C} and find the leftmost entry not equal to 0, let it be i . Let $j = \mathbb{C}[i]$, we exchange F_1 with F_j and do the following: (i) we swap the values in $\mathbb{S}''_{pre}[1]$ and $\mathbb{S}''_{pre}[j]$, then the values in $\mathbb{S}''_{suc}[1]$ and $\mathbb{S}''_{suc}[j]$; (ii) we set $\mathbb{S}''_{pre}[\mathbb{S}''_{suc}[j]] = j$ and $\mathbb{S}''_{suc}[\mathbb{S}''_{pre}[j]] = j$; (iii) finally, we set $\mathbb{S}''_{pre}[\mathbb{S}''_{suc}[1]] = 1$ and $\mathbb{S}''_{suc}[\mathbb{S}''_{pre}[1]] = 1$. Then, let $j' = \mathbb{S}''_{suc}[1]$. We exchange F_2 and $F_{j'}$ and then we make similar adjustments to their entries in \mathbb{S}''_{pre} and \mathbb{S}''_{suc} . We proceed in this fashion until we exhaust the linked list associated with the i th entry of \mathbb{C} . After that we continue to scan \mathbb{C} , find the leftmost non-zero entry $i' > i$ and process its associated list in the same way. At the end of the process, the F_i 's have been permuted in sorted stable order. Now both S'' and O are sorted. We merge them with the merging algorithm in [10] and S' is finally sorted.

5.3.2 Taking care of the encoded memory and the zones

In the last main step of the final phase we sort all those zones that have been built in the preliminary phase. We scan G , let $u \in G$ be the i th element accessed, we exchange $T_1[i]$, $T_2[i]$ and $T_3[i]$ with $J_1[u]$, $J_2[u]$ and $J_3[u]$, respectively. We swap S'' and H (H has been moved after S'' at the end of the aggregating phase). W , G , D , P , T_1 , T_2 , T_3 and H have $O(d) = O(n^\epsilon)$ elements and we can sort them using the mergesort in [10]. The obtained sequence can now be merged with S'' using the merging algorithm in [10]. J_1 , J_2 , J_3 and V have $O(n^\epsilon)$ elements. We sort them using the mergesort in [10]. Finally, M' and M'' have $\Theta(n/\log n)$ elements. We sort them with the mergesort in [10] and we are done.

Lemma 6. *The final phase requires $O(n)$ time, uses $O(1)$ auxiliary words and is stable.*

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